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SOME RECENT COMBINATORIAL APPLICATIONS OF BORSUK-TYPE THEOREMS

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1. INTRODUCTION

The well known theorem of Borsuk [Bo] is the following.

Theorem 1.1 (Borsuk)

For every continuous mapping $f: S^n \to R^n$, there is a point $x \in S^n$ such that $f(x) = f(-x)$. In particular, if f is antipodal (i.e. $f(x) = -f(-x)$ for all $x \in S^n$) then there is a point of S^n which maps into the origin.

This theorem and its many generalizations have numerous applications in various branches of mathematics, including Topology, Functional Analysis, Measure Theory, Differential Equations, Approximation Theory, Geometry, Convexity and Combinatorics. An extensive list of these applications, some of which are about fifty years old, appears in [Ste].

Most combinatorial applications of Borsuk's Theorem were found during the last ten years. The best known of these is undoubtfully Lovász's ingenious proof of the Kneser conjecture. Kneser [Kn] conjectured in 1955 that if $n \geq 2r + t - 1$ and all the r-subsets of an n-element set are colored by t colors then there are two disjoint r-sets having the same color. This was proved by Lovász twenty years later in [Lo]. Shortly afterwards, Bárány [Ba] gave a charming short proof. Both proofs apply Borsuk's theorem. In [BB], Bajmóczy and Bárány deduce an interesting generalization of Radon's Theorem from Theorem 1.1. Radon's Theorem states that for any linear map f from the $(n+1)$-dimensional simplex Δ^{n+1} to the n dimensional Euclidean space R^n, there are two disjoint faces of Δ^{n+1} whose images intersect. The authors of [BB] observed that this statement, for every *continuous* map f, follows easily from Borsuk's Theorem. A more general statement was proved by Bárány, Shlosman and Szűcs in [BSS]. They showed that for every prime p and every n, if $N = (p-1)(n+1)$ and $f: \Delta^N \to R^n$ is a continuous map, then there exist p pairwise disjoint faces of Δ^N, such that the intersection of all their images is nonempty. This generalizes (for prime p) a theorem of Tverberg [Tv], who proved the above for every linear map f, but without the assumption that p is a prime.

In order to establish their theorem, the authors of [BSS] proved the following interesting generalization of Borsuk's Theorem. For a prime k and for $m \geq 1$, let $X = X_{m,k}$ denote the CW-complex consisting of k disjoint copies of the $m(k-1)$ dimensional ball with an identified boundary $S^{m(k-1)-1}$. Define a free action of the cyclic group Z_k on X by defining w, the action of its generator as follows, (see [Bou], Chapter 13, for the definition of a free group action on a topological space). Represent $S^{m(k-1)-1}$ as the set of all m by k real matrices (a_{ij}) satisfying $\sum_{j=1}^{k} a_{ij} = 0$ for all $1 \leq i \leq m$ and $\sum_{i,j} a_{ij}^2 = 1$. Define now $w(a_{ij}) = (a_{i,j-1})$, where $j+1$ is reduced modulo k. Thus w just cyclically shifts the columns of a matrix representing a point of $S^{m(k-1)-1}$. Trivially, this action is free, i.e., $w(x) \neq x$ for all $x \in S^{m(k-1)-1}$. The map w is extended from $S^{m(k-1)-1}$ to $X_{m,k}$ as follows. Let (y,r,q) denote a point of X from the q-th ball with radius r and $S^{m(k-1)-1}$ - coordinate y. Then $w(y,r,q) = (wy,r,q+1)$, where $q+1$ is reduced modulo k. Since k is a prime, w defines a free Z_k action on $X = X_{m,k}$.

Theorem 1.2 ([BSS]).

For any continuous map $h: X \to R^m$ there exists an $x \in X$, such that $h(x) = h(wx) = \cdots = h(w^{k-1}x)$.

In Sections 3 and 4, we discuss some recent combinatorial applications of this theorem.

Another interesting application of Borsuk's Theorem was given by Bárány and Lovász in [BL]. They proved that the number of vertices of any centrally symmetric simple polytope in R^n is at least 2^n (which is the number of vertices of the n-cube). Very recently, R. Stanley [Sta] proved a more general result using other algebraic methods.

There are several other cominatorial applications of Theorem 1.1, including an interesting result of Yao and Yao [YY] in computational geometry. Some of these an be found in [Bj]. In the next three sections we discuss three additional, more recent examples. The first, proved in Section 2, is the following simple result of Akiyama and the present author. The case $d=2$ of this result is a well known Putman Problem (see, e.g. [La]).

Theorem 1.3 ([AA])

Let $A_1, A_2, \ldots, A_d$ be d pairwise disjoint subsets of R^d, each containing precisely n points, and suppose that the points in $A = \bigcup_{i=1}^{d} A_i$ are in general position, (i.e., no hyperplane contains $d+1$ of the points). Then there

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Our second example, discussed in Section 3, is the following.

Theorem 1.4 ([A1]).

Let N be an opened necklace with ka_i beads of color i, $1 \le i \le t$. Then it is possible to cut N in $(k-1)\cdot t$ places and partition the resulting intervals into k collections, each containing precisely a_i beads of color i, $1 \le i \le t$.

This theorem is best possible, and solves a problem of Goldberg and West [GW] (see also [AW]), who proved it for $k=2$. Its continuous analogue generalizes a theorem of Hobby and Rice [HR] on L_1-approximation.

In Section 4 we describe, very briefly, the proof of the following result, due to Frankl, Lovász and the present author, see [AFL].

Theorem 1.5 (The general Kneser problem)

If $n \ge (t-1)(k-1)+k\cdot r$ and all the r-subsets of an n-element set are colored by t colors then there are k pairwise disjoint r-sets having the same color.

This result is best possible and establishes a conjecture of Erdős [E], (see also [Gy]). For $k=2$ the statement of the theorem is Kneser conjecture mentioned above which was proved by Lovász. The case $r=2$ was proved by Cockayne and Lorimer [CL] and, independently, by Gyárfás [Gy]. The case $t=2$ was proved by Frankl and the present author in [AF].

Finally, in Section 5, we mention a few open problems.

2. DISJOINT SIMPLICES

As observed by Ulam, Borsuk's Theorem implies the following result, known under the self-explanatory name "the ham sandwich theorem."

Theorem 2.1

Let $\mu_1, \mu_2, \ldots, \mu_d$ be d probability measures on R^d, each absolutely continuous with respect to the usual Lebesgue measure. Then there exists a hyperplane H in R^d, which bisects all d measures, i.e., $\mu_i(H^+) = \mu_i(H^-) = \frac{1}{2}$ for all $1 \leq i \leq d$, where H^+ and H^- denote, respectively, the open positive side and the negative side of H.

Theorem 2.1 is usually deduced from Borsuk's Theorem as follows. One first shows, using measure-theoretic arguments, that for each unit vector $u \in S^{d-1}$ there is a hyperplane $H=H(u)$, perpendicular to u, with u oriented from H^- to H^+, which depends continuously on u and bisects μ_d, i.e., $\mu_d(H^+) = \mu_d(H^-)$. Next one defines a continuous function $f: S^{d-1} \rightarrow R^{d-1}$ by $f(v) = (\mu_1(H^+(v)), \ldots, \mu_{d-1}(H^+(v)))$. Since $H^+(v) = H^-(-v)$ the assertion of Theorem 2.1 now follows from that of Theorem 1.1.

We next apply the last theorem to prove the following.

Lemma 2.2

Let $A, A_1, A_2, \ldots, A_d$ be as in Theorem 1.3. Then there

$$|H^+ \cap A_i| = \lfloor n/2 \rfloor \text{ and } |H^- \cap A_i| = \lfloor n/2 \rfloor \text{ for all } 1 \leq i \leq d. \tag{2.1}$$

(Notice that if n is odd (2.1) implies that H contains precisely one point from each A_i.)

Proof.

Replace each point $p \in A$ by a ball of radius ϵ centered in p where ϵ is small enough to guarantee that no hyperplane intersects more than d balls. Associate each ball with a uniformly distributed measure of $1/n$. For $1 \leq i \leq d$ and a (lebesgue)- measurable subset T of R^d define $\mu_i(T)$ as the total measure of balls centered at points of A_i captured by T. Clearly $\mu_1, \mu_2, \ldots, \mu_d$ are continuous probability measure. By Theorem 2.1 there exists a hyperplane H in R^d such that $\mu_i(H^+) = \mu_i(H^-) = \frac{1}{2}$ for all $1 \leq i \leq d$. If n is odd, this implies that H intersects at least one ball centered at a point of A_i. However, H cannot intersect more than d balls altogether, and thus it intersects precisely one ball centered at a point of A_i, and it must bisect these d balls. Hence, for odd n, H satisfies (2.1). If n is even, H intersects at most d balls, and by slightly rotating H we can divide the centers of these balls between H^+ and H^- as we wish, without changing the position of each other point of A with respect to H. One

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We can now prove Theorem 1.3 by induction of n. For $n=1$ the result is trivial. Assuming the result for all $n', n' < n$, let $A, A_1, A_2, \ldots, A_d$ be as in Theorem 1.3 and let H be a hyperplane, guaranteed by Lemma 2.2, satisfying (2.1). Put $B_i = H^+ \cap A_i$ and $C_i = H^- \cap A_i$ for $1 \leq i \leq d$, $B = B_1 \cup \cdots \cup B_d$ and $C = C_1 \cup \cdots \cup C_d$. By the induction hypothesis, applied to $B, B_1, \ldots, B_d$ and to $C, C_1, \ldots, C_d$ we obtain two sets S_1 and S_2 of $\lfloor n/2 \rfloor$ pairwise disjoint simplices each, where each simplex of S_1 contains precisely one vertex from each B_i and each simplex of S_2 contains precisely one vertex from each C_i. Clearly, all the simplices in S_1 lie in H^+ and all those in S_2 lie in H^-.

We thus obtained $2 \cdot \lfloor n/2 \rfloor$ pairwise non-intersecting simplices. These, together with the simplex spanned by $A_i \cap H$ if n is odd, complete the induction and the proof of the theorem. □

3. SPLITTING NECKLACES

Let N be a necklace opened at the clasp with $k \cdot a_i$ beads of color i, $1 \leq i \leq t$. A *k-splitting* of the necklace is a partition of N into k parts, each consisting of a finite number of non-overlapping intervals of beads whose union captures precisely a_i beads of color i, $1 \leq i \leq t$. The *size* of the k-splitting is the number of cuts that form the intervals of the splitting. Thus, Theorem 1.4 simply asserts that every necklace with ka_i beads of color i, $1 \leq i \leq t$, has a k-splitting of size at most $(k-1) \cdot t$. One can easily check that the number $(k-1) \cdot t$ is best possible: indeed if the beads of each color appear contiguously on the opened necklace, then any k-splitting must contain at least $k-1$ cuts between the beads of each color, and hence its size is at least $(k-1) \cdot t$.

To prove Theorem 1.4 we need to formulate a continuous version of it.

Let $I = [0,1]$ be the unit interval. An *interval t-coloring* is a coloring of the points of I by t colors, such that for each $i, 1 \leq i \leq t$, the set of points colored i is (Lebesgue) measurable. Given such a coloring, a k-splitting of size r is a sequence of numbers $0 = y_0 \leq y_1 \leq \cdots \leq y_r \leq y_{r+1} = 1$ and a partition of the family of $r+1$ intervals $F = \{[y_i, y_{i+1}] : 0 \leq i \leq r\}$ into k pairwise disjoint subfamilies $F_1, \ldots, F_k$ whose union is F, such that for each $1 \leq j \leq k$ the union of the intervals in F_j captures precisely $1/k$ of the total measure of each of the t colors. Clearly, if each color appears contiguously and colors occupy disjoint intervals, the size of each k-splitting is at least $(k-1) \cdot t$. Therefore, the next theorem is best possible.

Theorem 3.1

Every interval t-coloring has a k-splitting of size $(k-1) \cdot t$

It is not difficult to check that this theorem implies Theorem 1.1. Indeed, given an opened necklace of $\sum_{i=1}^{t} ka_i = k \cdot n$ beads as in Theorem 1.4, convert it into an interval coloring by partitioning $I = [0,1]$ into $k \cdot n$ segments and coloring the j-th segment by the color of the j-th bead of the necklace. By Theorem 3.1 there is a k-splitting with at most $(k-1) \cdot t$ cuts, but these cuts need not occur at the endpoints of the $k \cdot n$ segments. One may now show, by induction on the number of "bad" cuts, that this splitting can be modified to form a k-splitting of the same size with no bad cuts, i.e., a splitting of the discrete necklace. The details are left to the reader.

Theorem 3.1 clearly follows from the following two assertions.

Proposition 3.2

Theorem 3.1 holds for every prime k.

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Proposition 3.3

The validity of Theorem 3.1 for (t,k) and for (t,l) implies its validity for $(t,k \cdot l)$.

The (easy) proof of Proposition 3.3 is left to the reader. To prove Proposition 3.2 we need the following additional result from [BSS].

Put $N = (k-1) \cdot (m+1)$ and let Δ^N denote the N-dimensional simplex, i.e., $\Delta^N = \{(x_0,x_1,\ldots,x_N) \in R^{N+1}, x_i \ge 0 \text{ and } \sum_{i=0}^{N} x_i = 1\}$. The *support* of a point $x \in \Delta^N$ is the minimal face of Δ^N that contains x. Let $Y = Y_{N,k}$ denote the following CW-complex;

$$Y_{N,k} = \{(y_1,y_2,\ldots,y_k): y_1,\ldots,y_k \in \Delta^N$$

and the supports of the $y_i - s$ are pairwise disjoint$\}$

There is an obvious free Z_k action on $Y_{N,k}$; its generator γ maps $(y_1,\ldots,y_k)$ into $(y_2,\ldots,y_k,y_1)$.

Let T and R be two topological spaces and suppose that Z_k acts freely on both. Let α and β denote the actions of the generator of Z_k on T and R, respectively. We say that a continuous mapping $f: T \to R$ is Z_k*-equivariant* if $f \circ \alpha = \beta \circ f$, (cf [Bou], Chapter 13).

Recall that for $s \ge 0$, a topological space T is s-*connected* if for all $0 \le \ell \le s$, every continuous mapping of the ℓ dimensional sphere S^ℓ into T can be extended to a continuous mapping of the $\ell+1$ dimensional ball $B^{\ell+1}$ with boundary S^ℓ into T.

Lemma 3.4 [BSS].

Suppose k is a prime, $m \ge 1, N = (k-1)(m+1)$ and let $X = X_{m,k}, Y = Y_{N,k}, \mu$ and γ be as in the preceding paragraphs. Then Y is $N-k = \dim X - 1$ connected and thus there is a Z_k-equivariant map $f: X \to Y$.

We can now prove Proposition 3.2. Let k be a prime and let c be an interval t-coloring. Put $X = X_{t-1,k}$, $Y = Y_{(k-1)t,k}$ and define a continuous function $g: Y \to R^{t-1}$ as follows. Let $y = (y_1,y_2,\ldots,y_k)$ be a point of Y. Recall that each y_i is a point of Δ^N, i.e., is an $N+1$ dimensional vector with nonnegative coordinates whose sum is 1, and that the supports of the $y_i - s$ are pairwise disjoint. Put $x = (x_0,x_1,\ldots,x_N) = \frac{1}{k}(y_1 + y_2 + \cdots + y_k)$, and define a partition of $[0,1]$ into $N+1$ intervals $I_0,I_1,\ldots,I_N$, where $I_0 = [0,x_0]$, $I_j = \left[\sum_{i=0}^{j-1} x_i, \sum_{i=0}^{j} x_i\right]$, $1 \le j \le N$. Notice that since the supports of the $y_i - s$ are pairwise disjoint, if $x_j > 0$ (i.e., the interval I_j has positive length), then there is a unique ℓ, $1 \le \ell \le k$ such that the j-th coordinate of y_ℓ is positive.

For $1 \leq \ell \leq k$, let F_ℓ be the family of all those I_j-s such that the j-th coordinate of y_ℓ is positive. Notice that the sum of lengths of these I_j-s is precisely $1/k$. For $1 \leq i \leq t-1$, define $g_i(y)$ to be the measure of the ith color in $\cup F_1$. Finally, put $g(y) = (g_1(y), g_2(y), \ldots, g_{t-1}(y))$. One can easily check that $g: Y \rightarrow R^{t-1}$ is continuous. Moreover, for $1 \leq \ell \leq k$ and $1 \leq i \leq t-1$, $g_i(\gamma^{\ell-1} y)$ is the measure of the ith color in $\cup F_\ell$. By Lemma 3.4 there exists a Z_k-equivariant map $f: X_{(t-1),k} \rightarrow Y_{(t-1)k,k}$. Define $h: gof: X \rightarrow R^{t-1}$. By Theorem 1.2 there is some $x \in X$ such that $h(x) = h(wx) = \cdots = h(w^{k-1}x)$. By the equivariance of f, $y = f(x)$ satisfies $g(y) = g(\gamma y) = \cdots = g(\gamma^{k-1} y)$. But this means that each of the families of intervals $F_1, F_2, \ldots, F_k$ corresponding to y captures precisely $1/k$ of the measure of each of the first $t-1$ colors. Since the total measure of each F_j is $1/k$, each F_j captures precisely $1/k$ of the measure of the last color, as well. Dividing the length 0 intervals arbitrarily between the F_j-s we conclude that there is a k-splitting of size $N = (k-1) \cdot t$, as desired. This completes the proof of Proposition 3.2.

Combining the methods of this Section with a simple compactness argument one can prove the following generalization of Theorem 3.1.

Theorem 3.5

Let $\mu_1, \mu_2, \ldots, \mu_t$ be t continuous probability measures on the unit interval. Then it is possible to cut the interval in $(k-1) \cdot t$ places and partition the $(k-1) \cdot t + 1$ resulting intervals into k families $F_1, F_2, \ldots, F_k$ such that $\mu_i(\cup F_j) = 1/k$ for all $1 \leq i \leq t$, $1 \leq j \leq k$. The number $(k-1) \cdot t$ is best possible.

The case $k=2$ of the last theorem is the Hobby-Rice theorem [HR] on L_1 approximation.

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4. THE GENERAL KNESER PROBLEM.

The basic ideas in the proof of Theorem 1.5 are similar to those used by Lovász in [Lo], but there are several additional complications. It is first useful to reformulate Theorem 1.5 in terms of the chromatic number of a Kneser hypergraph. Let $G = G_{n,k,r}$ be the k-uniform Kneser hypergraph defined as follows. The vertices of G are all the r-subsets of $\{1, 2, \dots, n\}$, and a collection of k vertices forms an edge if the corresponding r-sets are pairwise disjoint. Theorem 1.5 is thus equivalent to the statement that if $n \ge (t-1)(k-1) + k\cdot r$ then $G_{n,k,r}$ is not t-colorable.

For any k-uniform hypergraph $H = (V, E)$, define a simplicial complex, $C(H)$ as follows: the vertices of $C(H)$ are all the $|E|k!$ ordered k-tuples $(v_1, v_2, \dots, v_k)$ of vertices of H, where $\{v_1, \dots, v_k\} \in E$. A set of vertices $(v_1^i, \dots, v_k^i)_{i \in I}$ of $C(H)$ forms a simplex if there is a complete k-partite subgraph of H on the (pairwise disjoint) sets of vertices $V_1, V_2, \dots, V_k$ such that $v_j^i \in V_j$ for all $i \in I$ and $1 \le j \le k$.

Theorem 1.5 now follows from the following three assertions.

Proposition 4.1

For any k-uniform hypergraph H, where k is a prime, if $C(H)$ is $(t-1)(k-1)-1$ connected, then H is not t-colorable

Proposition 4.2

$C(G_{n,k,r})$ is $(n-kr-1)$-connected. Thus if $n \ge (t-1)(k-1) + kr$ it is $(t-1)(k-1)-1$-connected.

Proposition 4.3

The validity of Theorem 1.1 for (r, t, k) and $(r' = (t-1)(k-1)+kr, t, k')$ implies its validity for $(r, t, k\,k')$.

Proposition 4.1 appears interesting in its own right and probably holds for every positive integer k. Its proof uses the generalization of Borsuk Theorem due to Bárány, Shlosman and Szücs, given in Theorem 1.2. Proposition 4.2 can be proved using several standard results in topology and the (easy) proof of Proposition 4.3 is purely combinatorial. The detailed proofs appear in [AFL].

Propositions 4.1 and 4.2 imply the assertion of Theorem 1.5 for every prime k. Thus, by Proposition 4.3 the theorem holds for all r, t, k.

5. OPEN PROBLEMS

The first obvious problem is the problem of finding pure combinatorial proofs for the problems discussed in this paper. After all, one would naturally expect that combinatorial statements about combinatorial objects should have combinatorial proofs. Such proofs are desirable, since they might shed more light on the problems. At the moment, there is no known combinatorial proof to any of the combinatorial applications of Borsuk's theorem mentioned in this paper.

Another intriguing problem is an algorithmic one. When we use Borsuk's theorem to prove the existence of a certain partition, the proof supplies no practical way for effecting such a partition. Thus, for example, one would like to find a polynomial time algorithm for finding, given an opened necklace N with ka_i beads of color i, $1 \leq i \leq t$, a set of at most $(k-1)\cdot t$ cuts in N and a partition of the resulting intervals into k collections, each containing precisely a_i beads of color i, $1 \leq i \leq t$. It is worth noting that we can show that the following related problem is NP-complete: Given an opened necklace N with $2a_i$ beads of color i, $1 \leq i \leq t$, and given a set of cuts of N, decide if it is possible to divide the resulting intervals into two collections, each containing precisely a_i beads of color i, $a \leq i \leq t$.

Finally we mention another problem which is related to the results of Section 3. Suppose $\mu_1, \mu_2, \ldots, \mu_t$ are t probability measures on the unit inteval I, each absolutely continuous with respect to the usual (Lebesque) measure. For a real number α, $0 \leq \alpha \leq 1$ a subset A of I is an α-share (with respect to the measures $\mu_1, \ldots, \mu_t$) if $\mu_i(A) = \alpha$ for all $1 \leq i \leq t$. We note that Liapounoff Theorem ([Li], see also [NP] and [Da]) implies that for each $\mu_1, \ldots, \mu_t$ and α as above there is an α-share A. If A is a union of a finite number s of non-overlapping intervals we define the *size* of A to be s. Otherwise, the size of A is infinity. For an integer $t \geq 1$ and $0 \leq \alpha \leq 1$ let $f(t,\alpha)$ be the smallest integer f (possibly infinity) such that for every sequence of t continuous probability measures on I there is an α-share of size at most f. Clearly $f(t,0) = f(t,1) = 1$ for all $t \geq 1$ and $f(1,\alpha) = 1$ for all $0 \leq \alpha \leq 1$. The results of Stone and Tukey [ST] easily imply that $f(2,\alpha) = 1$ for every α of the form $1/k$, k integer, and that $f(2,\alpha) = 2$ for every other α. Combining Theorem 3.5 with an appropriate construction we can show that for every two integers $t,k \geq 1$.

$$f(t,1/k) = \left\lfloor \frac{t\cdot(k-1)+1}{k} \right\rfloor .$$

This implies that $f(t,\alpha)$ is finite for every rational α. It would be interesting to decide if $f(t,\alpha)$ is finite for all possible t and α and if so, to determine or estimate this function. At the moment, we are unable to show that $f(3,\alpha)$ is finite even for a single irrational value of α.

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